Research Statement

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1 Introduction

My research focuses on the area of Special Values of L-functions and their relations to the Weil-étale cohomology. One of the first result of special values of L-function is the Analytic Class Number Formula which I shall explain.

Let K be a number field with r_1 real embeddings and r_2 pairs of complex embeddings. The Dedekind zeta function associated with K is defined on the half plane $\mathrm{Re}(s) > 1$ as follows

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq 0} \frac{1}{N(\mathfrak{a})^s}$$

where the sum is taken over all non-zero integral ideals $\mathfrak{a}$ of K.

It is known that $\zeta_K(s)$ has a meromorphic continuation to $\mathbb{C}$ and has a simple pole at 1. The Analytic Class Number Formula is the following theorem

Theorem 1. *The leading term of the Laurent series expansion of $\zeta_K(s)$ at 1 is given by*

$$\zeta_K^*(1) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|\Delta_K|}w}$$

where h is the class number, R is the regulator , w is the number of roots of unity within K and Δ_K is the discriminant.

Using the functional equation of $\zeta_K(s)$, we can deduce that the order of vanishing of $\zeta_K(s)$ at 0 is $r_1 + r_2 - 1$ and we also obtain

Theorem 2. *The leading term of the Laurent series expansion of $\zeta_K(s)$ at 0 is given by*

$$\zeta_K^*(0) = -\frac{hR}{w}$$

These fundamental results have been generalized in many different directions and are still inspirations for recent works. One such instance is the work of Lichtenbaum. Let X be the spectrum of the ring of integers O_K of K and j be the natural map from $Spec(K)$ to X. In [L2], Lichtenbaum constructs a Grothendieck topology on X called the Weil-étale topology. He defines an Euler characteristic $\chi(\mathbb{Z})$ in terms of the Weil-étale cohomology groups with compact support $H^n_W(\bar{X}, \phi_!\mathbb{Z})$ and shows that

$$\chi(\mathbb{Z}) = \pm\zeta_K^*(0)$$

hence gives the first evidence for the philosophy that special values of L-functions are Euler characteristics of a complex of Weil-étale cohomology groups.

2 My Thesis

Let K be a number field with Galois group G_K. Let T be an algebraic torus over K. The character group $\hat{T}$ of T is a torsion free, finitely generated integral representation of G_K. For each non-zero prime ideal $\mathfrak{p}$ of K, let $F_\mathfrak{p}$ be the inverse of the Frobenius map of K at $\mathfrak{p}$ and $I_\mathfrak{p}$ be the inertia subgroup of G_K at $\mathfrak{p}$. On the half plane $\mathrm{Re}(s) > 1$, the L-function associated to T is defined as

$$L(\hat{T}, s) = \prod_{\mathfrak{p}} L_p(\hat{T}, s) = \prod_{\mathfrak{p}} \det(Id - F_\mathfrak{p} N(\mathfrak{p})^{-s} | \hat{T}_\mathbb{Q}^{I_\mathfrak{p}})^{-1}$$

where the product is taken over all non-zero prime ideal $\mathfrak{p}$ of K and $N(\mathfrak{p})$ denote the norm of $\mathfrak{p}$. It is known that $L(\hat{T}, s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$, has a pole at 1 of order equal to the rank of $H^0(K, \hat{T})$ and vanishes at 0 with order of vanishing equals $\sum_{v \in S_\infty} \mathrm{rank}_\mathbb{Z} H^0(K_v, \hat{T}) - \mathrm{rank}_\mathbb{Z} H^0(K, \hat{T})$ ([T1]) where K_v is the completion of K at v.

In my thesis, I study the relationship between the special values of L-function of an algebraic torus and the Weil-étale cohomology. Unfortunately, I am unable to construct a Weil-étale topology that is suited for the purpose of studying the special values of L-functions. Therefore I have to define the Weil-étale cohomology groups in terms of the étale cohomology groups. Let us write $(j_*\hat{T})^D$ for the complex of étale sheaves $R\mathcal{H}om(j_*\hat{T}, \mathbb{G}_m)[-1]$. We make the following definitions :

Definition 3.

$$H^n_W(X, j_*\hat{T}) = \begin{cases} H^n_{et}(X, j_*\hat{T}) & n = 0, 1 \\ h^n RHom_\mathbb{Z}(R\Gamma_{et}(X, j_*\hat{T}^D), \mathbb{Z}[-3]) & n > 1 \end{cases} \tag{1}$$

$$H^n_W(X, j_*\hat{T}^D) = \begin{cases} H^n_{et}(X, j_*\hat{T}^D) = \mathrm{Ext}^{n-1}_X(j_*\hat{T}, \mathbb{G}_m) & n = 0, 1 \\ h^n RHom_\mathbb{Z}(R\Gamma_{et}(X, j_*\hat{T}), \mathbb{Z}[-3])) & n > 1 \end{cases} \tag{2}$$

This definition is motivated by a duality of Lichtenbaum for Weil-étale cohomology of curves over finite field ([L1]). Having defined $H^n_W(X, j_*\hat{T})$ and $H^n_W(X, j_*\hat{T}^D)$, I can construct the Euler characteristic $\chi(j_*\hat{T})$ and $\chi(j_*\hat{T}^D)$. Using the techniques developed by Lichtenbaum and Bienenfeld in their unpublished work ([LB]), I show $L^*(\hat{T}, 0) = \pm\chi(j_*\hat{T})$ and $L^*(\hat{T}, 1) = \pm\chi(j_*\hat{T}^D)$. By computing $\chi(j_*\hat{T})$ and $\chi(j_*\hat{T}^D)$, I obtain the following result :

Theorem 4. *Let K be a totally imaginary number field and T be an algebraic torus over K with character group $\hat{T}$. Let $Ш^1(T)$ be the Tate-Shafarevich group of T. Then the leading term of the Laurent expansion of $L(\hat{T}, s)$ at 0 is given by*

$$L^*(\hat{T}, 0) = \pm \frac{h_T R_T}{w_T} \frac{[Ш^1(T)]}{[H^1(K, \hat{T})]} \prod_{\mathfrak{p} \notin S_\infty} [H^0(\hat{\mathbb{Z}}, H^1(I_\mathfrak{p}, \hat{T}))]$$

where h_T, R_T and w_T are the class number, regulator and the number of roots of unity of $T(K)$ (for the definitions of these terms, see [O1]).

and recover the following theorem of Ono ([O1] and [O2]) :

Theorem 5. *Let T be an algebraic torus of dimension d over a totally imaginary number field K. Let ω be a differential d-form of $T(K)$. Then the leading term of the Laurent expansion of $L(\hat{T}, s)$ at 1 is given by*

$$L^*(\hat{T}, 1) = \pm \frac{h_T R_T}{w_T} \frac{[\text{Ш}^1(T)]}{[H^1(K, \hat{T})]} \frac{\prod_{v \in S_\infty} \int_{T_v^c} \omega_v}{|\Delta_K|^{d/2}} \prod_{p \notin S_\infty} \int_{T_p^c} L_p(\hat{T}, 1)\omega_p$$

where T_v^c and T_p^c are the maximal compact subgroups of $T(K_v)$ and $T(K_p)$ respectively.

3 Future Research

3.1 Short Term Projects

Let A be an abelian variety of dimension d over a number field K and A^t be its dual abelian variety. Let $\mathcal{A}$ and $\mathcal{A}^t$ be the the Néron models of A and of A^t respectively. Fix a prime number l, let $T_l(A)$ be the Tate module of A at l. The L-function of A is defined on the half plane $\mathrm{Re}(s) > 3/2$ as follows

$$L(A, s) := \prod_p L_p(A, s) = \prod_p \det(Id - F_p N(p)^{-s} | Hom(T_l(A), \mathbb{Q}_l)^{I_p})^{-1}$$

It is known that this definition does not depend on l. The Mordell-Weil theorem asserts that the group of K-rational points $A(K)$ is finitely generated. Let us recall the celebrated Birch & Swinnerton-Dyer conjecture.

Conjecture 6 (Birch and Swinnerton-Dyer)**.** *Assume*

1. *The L-function $L(A, s)$ has an analytic continuation to a neighborhood of 1 in the complex plane.*
2. *The Tate-Shafarevich group $\text{Ш}(A)$ is finite.*

Then the order of vanishing of $L(A, s)$ at 1 is the rank of $A(K)$ and the leading term of the Laurent expansion of $L(A, s)$ at 1 is given by

$$L^*(A, 1) = \frac{[\text{Ш}(A)] R_A \Omega_A}{[A(K)_{tor}][A^t(K)_{tor}]|\Delta_K|^{d/2}} \prod_{v \notin S_\infty} c_v \tag{3}$$

where R_A is the determinant of the height pairing, Ω_A is the real period and c_v is the number of connected components of $\mathcal{A}(k_v)$, the reduction mod v of $\mathcal{A}$.

My immediate research plan is to apply the same method employed in my thesis to the special values of L-function of abelian varieties. We know that j_*A and j_*A^t are étale sheaves on X represented by $\mathcal{A}$ and $\mathcal{A}^t$ respectively. We make the following definition :

Definition 7.

$$H^n_W(X, j_*A[-1]) = \begin{cases} H^0_{et}(X, j_*A[-1]) & n = 0, 1 \\ h^n RHom_{\mathbb{Z}}(R\Gamma_{et}(X, j_*A^t[-1]), \mathbb{Z}[-3])) & n > 1 \end{cases} \tag{4}$$

My aim is to construct an Euler characteristic $\chi(j_*A[-1])$ and show that $\chi(j_*A[-1])$ equals the right hand side of (3). Thus the Birch and Swinnerton-Dyer conjecture implies $L^*(A,1) = \pm\chi(j_*A[-1])$.

Once this is achieved, I would like to extend these results to special values of L-functions of 1-motives. I have been working on 1-motives of the forms $[N \to T]$ where N is a torsion free, finitely generated G_K-module and T is an algebraic torus over K.

3.2 Long Term Goals

Special Values of L-functions is a vast area of algebraic number theory and arithmetic geometry with many open problems. One such open problem is the Tamagawa Number Conjecture which is originated by Bloch and Kato and refined by Fontaine and Perrin-Riou ([BK],[Fo]). I would like to study the connection between this conjecture and the Weil-étale cohomology approach to special values of L-function. One instance of such connection is the fact that theorem 4 is predicted by the Tamagawa Number Conjecture ([Z1]).

Another direction which I am planning to pursue is to replace the ad hoc definition 3 by a more natural definition, that is to really construct a Grothendieck topology as done by Lichtenbaum in [L2]. Furthermore, I would like also to study the relationship between special values of incomplete L-function and Weil-étale cohomology of sheaves defined only on a open subscheme of the spectrum of the ring of integers.

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